Problem Set 3

Due: **10:00pm, Friday, 7 February**

This problem set focuses on understanding uncountability (Chapter 2 in TCS) and introducing the Boolean circuit model for finite computation (Chapter 3 in TCS). Write your answers in the ps3.tex LaTeX template. You will submit your solutions in GradeScope as a PDF file with your answers to the questions in this template.

Collaboration Policy: You may discuss the problems with anyone you want. You are permitted to use any resources you find for this assignment **other than solutions from previous/concurrent CS3120 courses**. You should write up your own solutions and understand everything in them, and submit only your own work. You should note in the *Collaborators and Resources* box below the people you collaborated with and any external resources you used (you do not need to list resources you used for help with LaTeX).

Collaborators and Resources: TODO: replace this with your collaborators and resources (if you did not have any, replace this with *None*)

To do this assignment:

1. Open this read-only Overleaf project located at https://www.overleaf.com/read/ncsjqgfrjfys#f36733, then select the "Menu" button at the top-left, and then select "Copy Project". You will have an opportunity to rename the project, and then Overleaf will create a new copy of the project which you can edit.

2. Open your copy of the project and in the left side of the browser, you should see a file directory containing ps3.tex. Click on ps3.tex to see the LaTeX source for this file, and enter your solutions in the marked places. (You will also see the uvatoc.sty file, a "style" file that defines some useful macros. You are welcome to look at this file but should not need to modify it.)

3. The first thing you should do in ps3.tex is set up your name as the author of the submission by replacing the line, `\submitter{TODO: your name}`, with your name and UVA id, e.g., `\submitter{Haolin Liu (srs8rh)}`.

4. Write insightful and clear answers to all of the questions in the marked spaces provided.

5. Before submitting your ps3.pdf file, also remember to:
   - List your collaborators and resources, replacing the TODO in `\collaborators{TODO: replace ...}` with your collaborators and resources. (Remember to update this before submitting if you work with more people.)
   - Replace the second line in ps3.tex, `\usepackage{uvatoc}` with `\usepackage[response]{uvatoc}` so the directions do not appear in your final PDF.

Problem 1 *Map lists of integers to a number.*

For every set S, the set S^* is defined as the set of all finite sequences of members of S (i.e., $S^* = \{(x_0, ..., x_{n-1}) \mid n \in \mathbb{N}, \forall_{i\in[n]} x_i \in S\}$). Prove that $|\mathbb{Z}^*| \leq |\mathbb{N}|$ where $\mathbb{Z}$ is the set of all integers $\{..., -3, -2, -1, 0, +1, +2, +3, ...\}$. That is, to show a surjective function from $\mathbb{N}$ to $\mathbb{Z}^*$.

Note: It is easy to show that $|\mathbb{N}| \leq |\mathbb{Z}^*|$, and together with the above, it follows that $|\mathbb{Z}^*| = |\mathbb{N}|$ by (Cantor-)Schröder–Bernstein theorem.

Answer:

Problem 2 *Countable Graphs:* The book defines an *undirected graph* (Definition 1.3). We modify this by adding one word to define an *undirected finite graph* as follows:

Definition 1 (Undirected finite graph) An undirected finite graph $G = (V, E)$ consists of a *finite* set V of vertices and a set E of edges. Every edge is a size-two subset of V.

Prove that the set of all undirected finite graphs is *countably infinite*.

Answer:

Problem 3 *Maximum number of Inputs*

See the definition of Boolean circuit from the text book (Section 3.3).[1] The *depth* of a circuit is the length of the longest path (in the number of gates) from the an input to an output in the circuit. We say that an output y_j depends on an input x_i, if there are two input sequences that *only* differ in x_i and lead to different values in y_j. Suppose a Boolean circuit C has n inputs $x_1, \ldots, x_n$, has depth d for some $d \geq 0$, has only one output y, and that y depends on *all* of the n. Prove $n \leq 2^d$.

Note: there are different ways to prove this, but we recommend using induction.

Answer:

[1] See https://introtcs.org/public/lec_03_computation.html#booleancircuitsec

Problem 4 *Compare 4 bit numbers*

Draw a Boolean circuit or type a equivalent straightline program (using only *AND*, *OR*, and *NOT* gates) that computes the function $CMP_8 : \{0,1\}^8 \to \{0,1\}$ such that $CMP_8(a_0, a_1, a_2, a_3, b_0, b_1, b_2, b_3) = 1$ if and only if the number represented by $a_0a_1a_2a_3$ is larger than the number represented by $b_0b_1b_2b_3$. We will say that a_0, b_0 are the most significant bits and a_3, b_3 are least significant.

Answer:

Problem 5 *Compare n bit numbers*

Prove that there exists a constant c such that for every n there is a Boolean circuit (using only *AND*, *OR*, and *NOT* gates) C of at most $c \cdot n$ gates that computes the function $CMP_{2n} : \{0,1\}^{2n} \to \{0,1\}$ such that $CMP_{2n}(a_0 \cdots a_{n-1} b_0 \cdots b_{n-1}) = 1$ if and only if the number represented by $a_0 \cdots a_{n-1}$ is larger than the number represented by $b_0 \cdots b_{n-1}$.

In other words, generalize the previous problem to describe how to compare n-bit numbers for any specific value n using *AND*, *OR*, and *NOT* in such a way that the total number of gates used is $O(n)$ (i.e., asymptotically linear).

Answer:

Problem 6 *OR, NOT is universal*

Prove that the set $\{OR, NOT\}$ is universal, in the sense that one can compute NAND using these gates

Answer:

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This is the end of the problems for PS3. Remember to follow the last step in the directions on the first page to prepare your PDF for submission.