

HW 0 will be posted today and
due next Friday, Jan 23 (10:00pm)

Quiz 0 is posted,
due next Monday 10:00pm



Class 1: Definitions

University of Virginia
CS3120: DMT2

<https://weikailin.github.io/cs3120-toc>

Wei-Kai Lin

Recap:

Course goal:

To understand the power and limitations of computation

Ex. Fast multiplication

Def, rep
Karatsuba
analysis

$$\begin{array}{r|l|l} & \bar{x} & x \\ \times & \bar{y} & y \\ \hline & (\bar{x} + \underline{x})(\bar{y} + \underline{y}) & \underline{x}\underline{y} \\ + & \bar{x}\underline{y} & -\bar{x}\underline{y} - \underline{x}\underline{y} \\ \hline \bar{x}\underline{y} & (\bar{x}\underline{y} + \underline{x}\underline{y}) & \underline{x}\underline{y} \end{array}$$

(Fun) History of Math

Class 0

How did
Babylonian number
system work?

Place-value number system

- Babylonian (2000BC)




𐎶 1	𐎵 11	𐎶𐎵 21	𐎶𐎶 31	𐎶𐎶𐎵 41	𐎶𐎶𐎶 51
𐎶𐎶 2	𐎶𐎶𐎵 12	𐎶𐎶𐎶 22	𐎶𐎶𐎶𐎵 32	𐎶𐎶𐎶𐎶 42	𐎶𐎶𐎶𐎶𐎵 52
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- The number: $x_k x_{k-1} x_{k-2} \dots x_2 x_1 x_0$



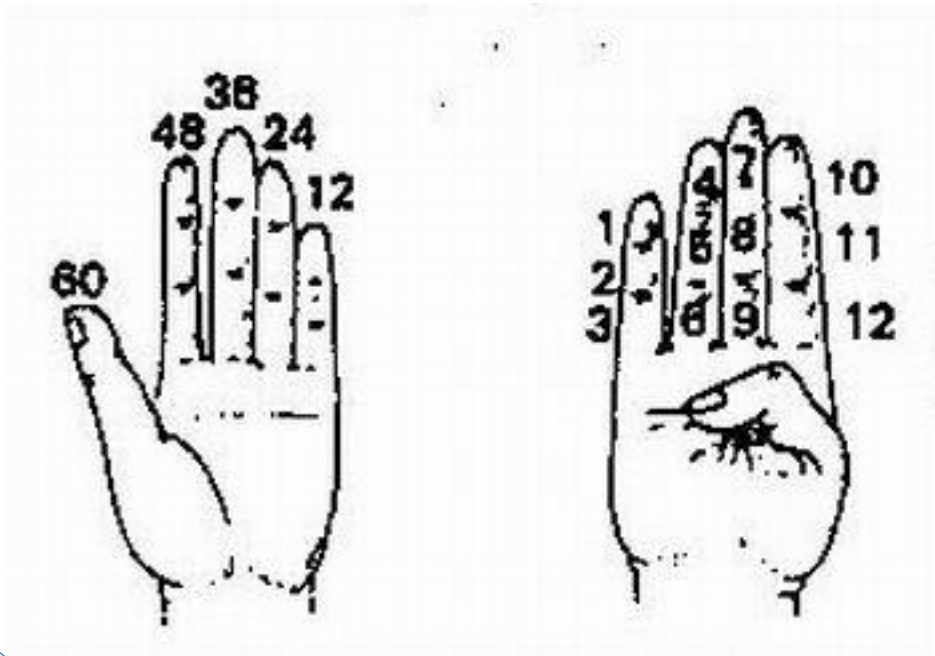
Means: $\sum x_i \cdot b^i$

- Extremely important development in history!

Babylonian Counting



𐎶 1	𐎶𐎵 11	𐎶𐎵𐎶 21	𐎶𐎵𐎶𐎵 31	𐎶𐎵𐎶𐎵𐎶 41	𐎶𐎵𐎶𐎵𐎶𐎵 51
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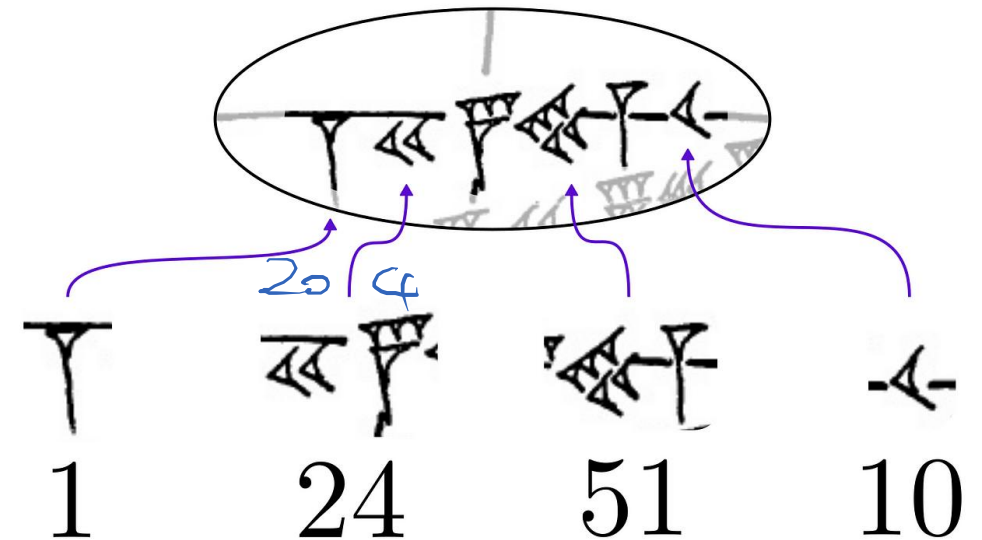


YBC 7289

Babylonian clay tablet from
~1700 BCE

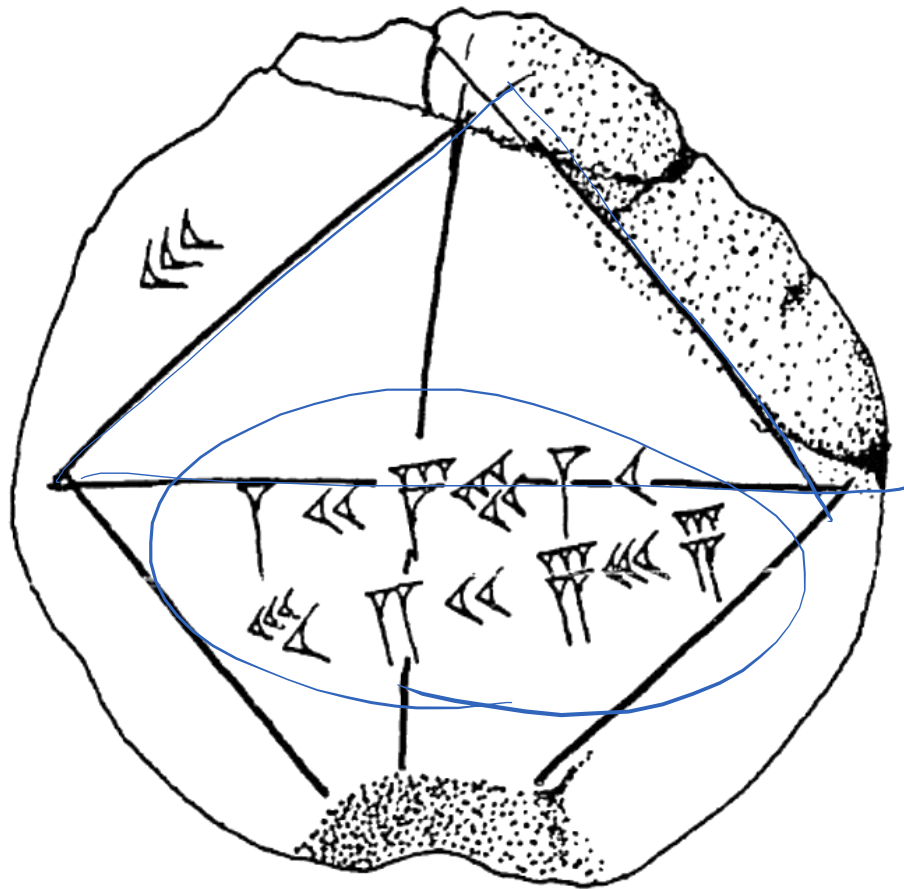
YBC 7289

Babylonian clay tablet from
~1700 BCE



YBC 7289

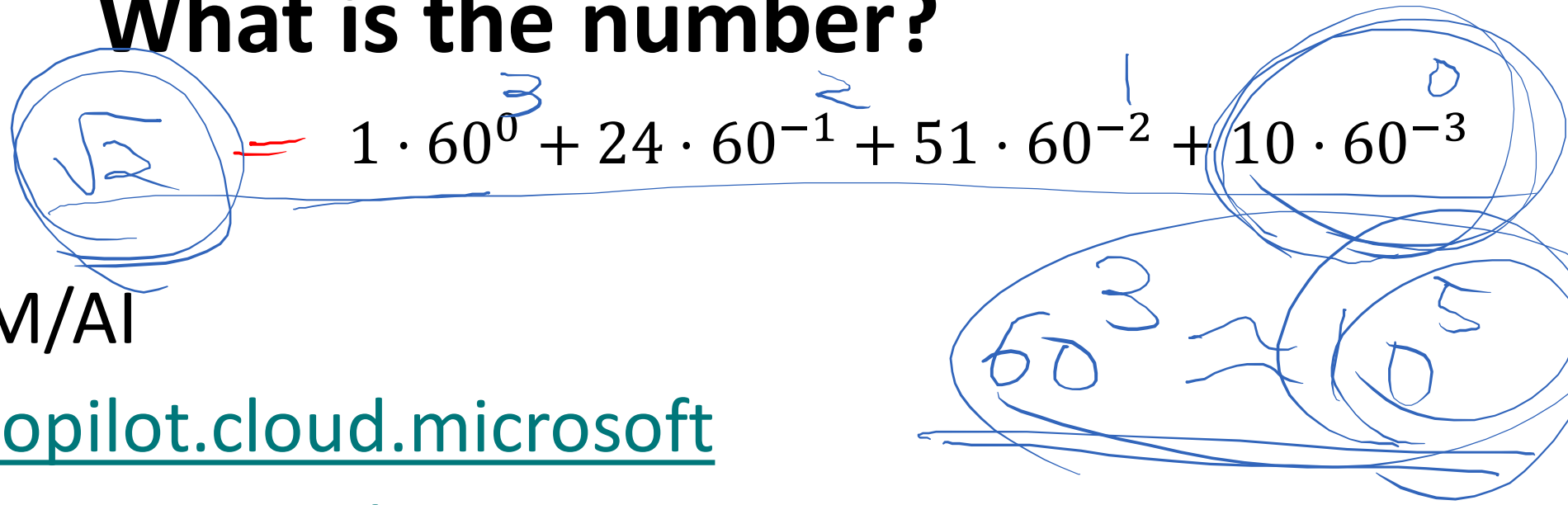
Babylonian clay tablet from
~1700 BCE



1	24	51	10

$$= 1 \cdot 60^0 + 24 \cdot 60^{-1} + 51 \cdot 60^{-2} + 10 \cdot 60^{-3}$$

What is the number?



Handwritten diagram showing a circle with a square root symbol $\sqrt{}$ and a red arrow pointing to the expression:

$$1 \cdot 60^{\overset{3}{0}} + 24 \cdot 60^{\overset{2}{-1}} + 51 \cdot 60^{\overset{1}{-2}} + 10 \cdot 60^{\overset{0}{-3}}$$

Below the expression, there are handwritten notes: $60^3 \approx 216,000$ and 10^5 .

Let's ask LLM/AI

- <https://copilot.cloud.microsoft>
- <https://gemini.google.com>

Plan

Mathematical Definitions

Today: Chapter 0 and 1 in the TCS book

What makes a good definition?

Defining the Natural Numbers

Induction Principle

Sets

Defining the binary strings

Quiz 0 (Survey and Pledge)

- Posted on Canvas and Gradescope.
- Let me know if you need access permission.

Homework 0

- You can use any resources you want
- You can discuss problems with anyone you want, but must write up solutions yourself and understand what you write
- You must credit the resources and people

Homework is about understanding the concepts and learning how to solve problems – you will need to demonstrate this on your own on the exams

Office Hours Start Next Monday!

Time and location:

<https://weikailin.github.io/cs3120-toc/calendar/>



***Is 0 a Natural
Number?***

Q3.3 Is 0 a Natural Number?

1 Point

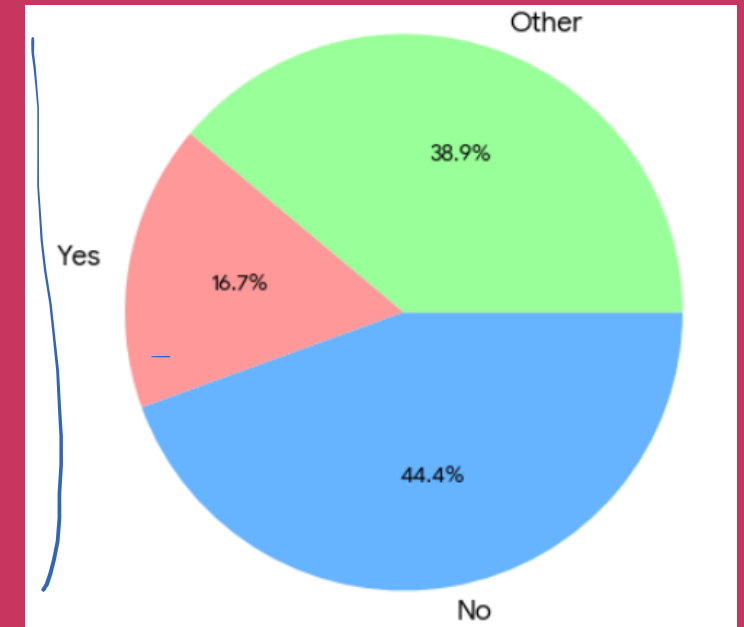
- ☐ Yes
- ☐ No
- ☐ We don't know.
- ☐ What's that?

Num

3

8

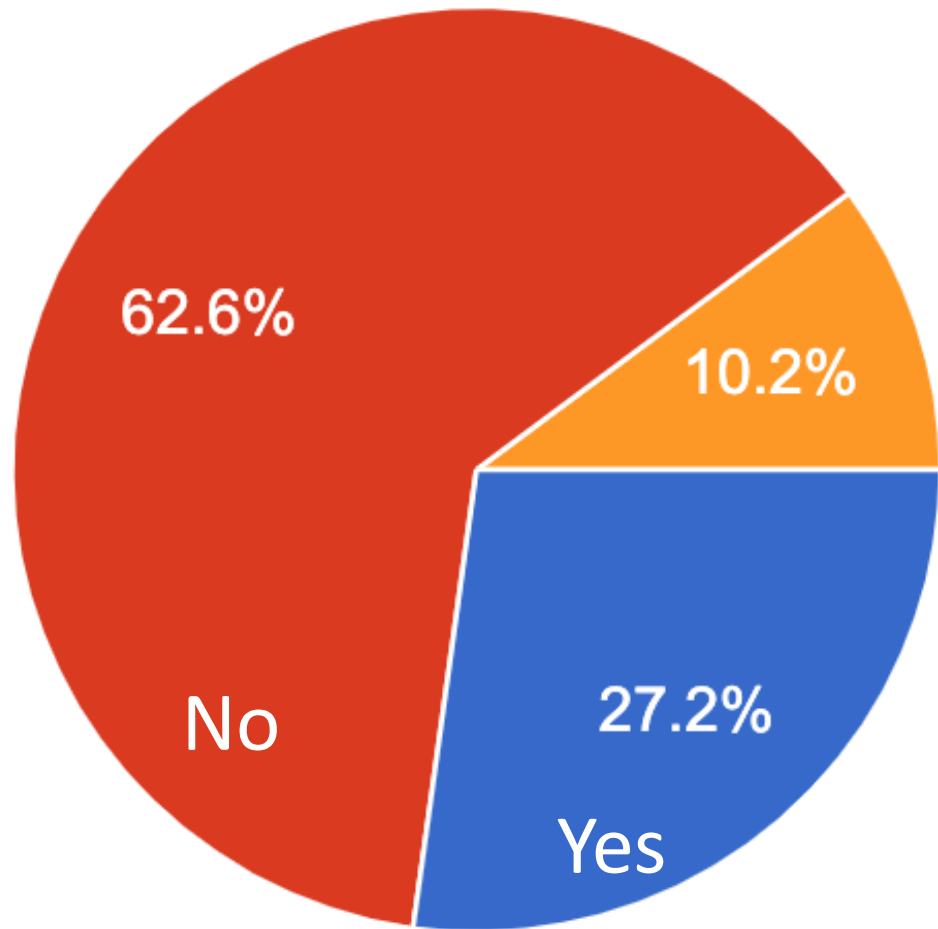
7



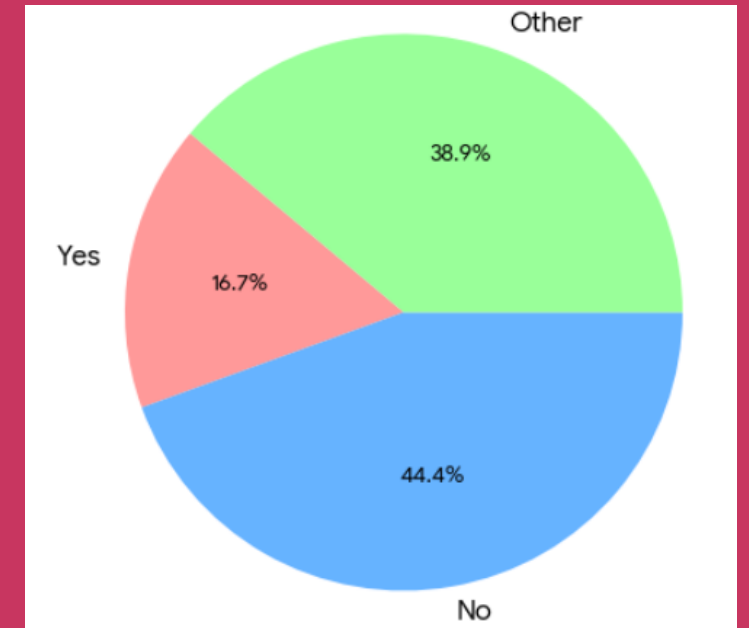
Depends on the class: in DMT1 it is a natural number, but in Basic Real Analysis it is not. It is preference for tradition or not.

Is 0 a Natural Number?

Responses from Fall 2019 Course:



Huh?



What makes a good definition?



SINCE 1828

Dictionary



definition

Full Definition of DEFINITION

- 1 : an act of **determining**; *specifically* : the formal proclamation of a Roman Catholic **dogma**
- 2 **a** : a statement expressing the essential nature of something
b : a statement of the meaning of a word or word group or a sign or symbol <dictionary *definitions*>
c : a product of **defining**
- 3 : the action or process of stating the meaning of a word or word group
- 4 **a** : the action or the power of describing, explaining, or making **definite** and clear <the *definition* of a telescope> <her comic genius is beyond *definition*>
b (1) : clarity of visual presentation : **distinctness** of outline or detail <improve the *definition* of an image> (2) : clarity especially of musical sound in reproduction
c : sharp **demarcation** of outlines or limits <a jacket with distinct waist *definition*>

What makes a good definition?

precise

concise (no extra info)

proven (principle)

What makes a good definition?

What makes a good definition?



A good definition has several key characteristics:

1. **Clarity:** It should be clear and easy to understand, avoiding ambiguous or vague language.
2. **Precision:** It should be precise, providing an exact explanation of the term without unnecessary details.
3. **Conciseness:** It should be concise, using as few words as possible while still conveying the full meaning.
4. **Comprehensiveness:** It should cover all essential aspects of the term, ensuring that the definition is complete.
5. **Non-circularity:** It should avoid using the term being defined within the definition itself.
6. **Contextual Relevance:** It should be relevant to the context in which the term is used, providing appropriate examples if necessary.

Would you like an example of a well-defined term?

Toy Example: Define “Bread”

- Ask LLM

can you give me a definition of "bread"



Copilot

Certainly! Here is a definition of "bread":

Bread (noun):

1. A food made from flour, water, and usually yeast, mixed together and baked. It is a staple food in many cultures and comes in various forms such as loaves, rolls, and flatbreads ¹.
2. In slang, it can also refer to money ¹.

For example:

- "She bakes bread every day."
- "I need to earn some bread to pay the bills."

If you have any other questions or need more information, feel free to ask!

Toy Example: Define “Bread”

- Dictionary: made from flour, water, and usually yeast, mixed together and baked



Chinese

Descriptive Definitions

Basic Analysis I

Introduction to Real Analysis. Volume I

Example 0.3.3: The following are sets including the standard notations.

(i) The set of *natural numbers*, $\mathbb{N} := \{1, 2, 3, \dots\}$.

by Jiří Lebl

June 8, 2021
(version 5.4)

https://ocw.mit.edu/courses/18-100a-real-analysis-fall-2020/resources/mit18_100af20_basic_analysis/

Descriptive Definitions

Basic Analysis I

Introduction to Real Analysis. Volume I

Example 0.3.3: The following are sets including the standard notations.

(i) The set of *natural numbers*, $\mathbb{N} := \{1, 2, 3, \dots\}$.

That the notation $|A| = n$ is justified we leave as an exercise. That is, for each nonempty finite set A , there exists a unique *natural number* n such that there exists a bijection from A to $\{1, 2, 3, \dots, n\}$.

Descriptive Definitions

Wikipedia screenshot, 2026

☰ Natural number

🌐 150 languages ▾

Article [Talk](#)

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From Wikipedia, the free encyclopedia



In [mathematics](#), the **natural numbers** are the numbers [0](#), [1](#), [2](#), [3](#), and so on, possibly excluding 0.^{[\[a\]](#)[\[1\]](#)} The terms **positive integers**, **non-negative integers**, **whole numbers**, and **counting numbers** are also used.^{[\[2\]](#)[\[3\]](#)} The [set](#) of the natural numbers is commonly denoted by a bold **N** or a [blackboard bold](#) **N**.

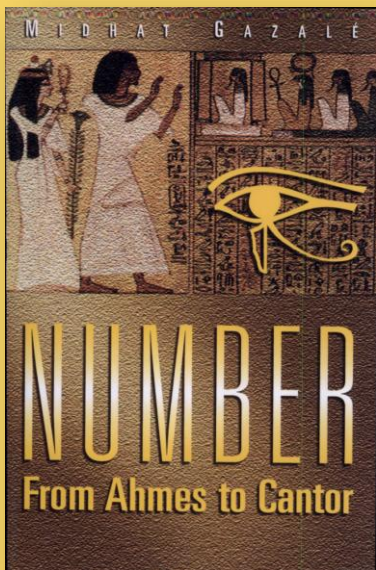
The natural numbers are used for counting, and for labeling the result of a count, such as: "there are *seven* days in a week", in which case they are



Natural numbers can be...

Peter Kahn, *The Natural Numbers*

So-called counting numbers have been used, with varied symbolic representation, since prehistoric times. Notation for these has stabilized to the standard Arabic numerals 1, 2, 3, ... for the past millennium or so, and classical usage has referred to these numbers as “natural.” The concept of zero achieved acceptance as a number much more recently—in roughly 300 B.C.E.—and its common denotation as 0 occurred well after the early usage of 1, 2, 3, So, it may be deemed to be slightly less natural than the others. Nevertheless, we (and most contemporary mathematicians) adjoin 0 to the counting numbers, call the totality $\{0, 1, 2, 3, \dots\}$ the *set of natural numbers*, and denote it by the letter \mathbb{N} .



Definition-by-Example

If you are asked the question “What is a natural number?” you will probably reply, “1, 2, 3, . . . ,” and not be aware that your reply was not to the question originally put to you, but to the instruction “Recite the natural numbers.” In so doing, you have, unbeknownst to you, revealed your affiliation with the intuitionist school of the ancient Greeks, who believed that numbers, which were the essence of everything, just were! To the ancient Greeks, numbers were god-given, and to recite them was to define them.

Definition-by-Example

TCS Book

1.4.2 Special sets

There are several sets that we will use in this book time and again. The set

$$\mathbb{N} = \{0, 1, 2, \dots\}$$

contains all *natural numbers*, i.e., non-negative integers. For any natural number $n \in \mathbb{N}$, we define the

The symbol \mathbb{N} stands for the set of *nonnegative integers*: 0, 1, 2, 3, ... (ask your instructor for the complete list).

10 ... 100 ...

Can Wei-Kai (or any human) show the list?

NO

The symbol \mathbb{N} stands for the set of *nonnegative integers*: 0, 1, 2, 3, ... (ask your instructor for the complete list).

1.4.2 Special sets

TCS Book

There are several sets that we will use in this book time and again. The set

$$\mathbb{N} = \{0, 1, 2, \dots\}$$

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Definition. A *Natural Number* is a non-negative integer.

Definition. A *Natural Number* is a non-negative integer.

We will also occasionally use the set $\mathbb{Z} = \{\dots, -2, -1, 0, +1, +2, \dots\}$ of (negative and non-negative) *integers*,¹ as well as the set \mathbb{R} of *real*

¹ The letter Z stands for the German word “Zahlen”, which means *numbers*.

TCS

The **set** of integers consists of zero (**0**), the positive **natural numbers** (**1, 2, 3, ...**), also called *whole numbers* or *counting numbers*,^{[2][3]} and their **additive inverses** (the **negative integers**, i.e., **-1, -2, -3, ...**).

Wikipedia

*How can we define something **new***
(more complex than things we already know)?

Constructive Definitions

Two symbols:

0

“Zero”

S

“*Successor*”

Constructive Definitions

Two symbols:

0 “Zero”

S “*Successor*”

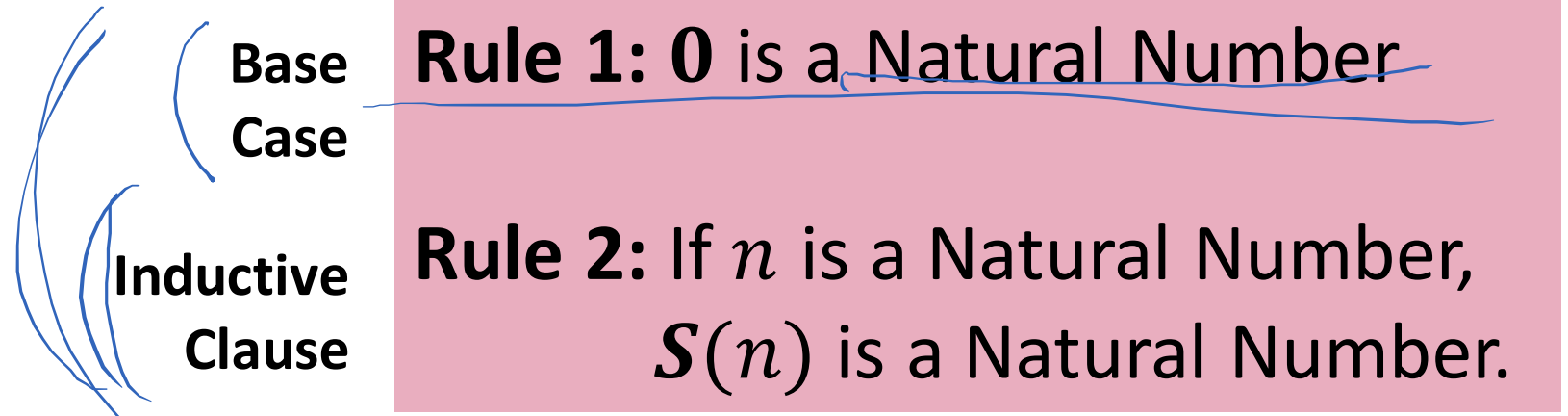
Constructive definition of the ***Natural Numbers***:

Rule 1: 0 is a Natural Number

Rule 2: If n is a Natural Number,
 $S(n)$ is a Natural Number.

Constructive Definitions

Constructive definition of the Natural Numbers:



Recursive definition: we use Natural Number in our definition?!

// ARITHMETICES PRINCIPIA //

NOVA METHODO EXPOSITA

888
6

A

IOSEPH PEANO

in R. Academia militari professore

Analysin infinitorum in R. Taurinensi Athenaeo docente.



AUGUSTAE TAURINORUM
EDIDERUNT FRATRES BOCCA
REGIS BIBLIOPOLAE

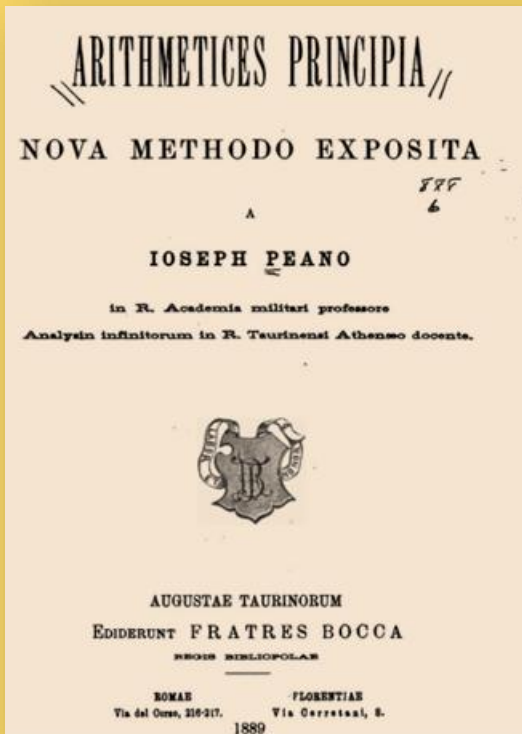
ROMAE
Via del Corso, 216-217.

FLORENTIAE
Via Cerretani, 8.

1889



Giuseppe Peano
1858-1932



ARITHMETICES PRINCIPIA.

§ 1. De numeris et de additione.

Explicationes.

Signo N significatur *numerus (integer positivus)*.

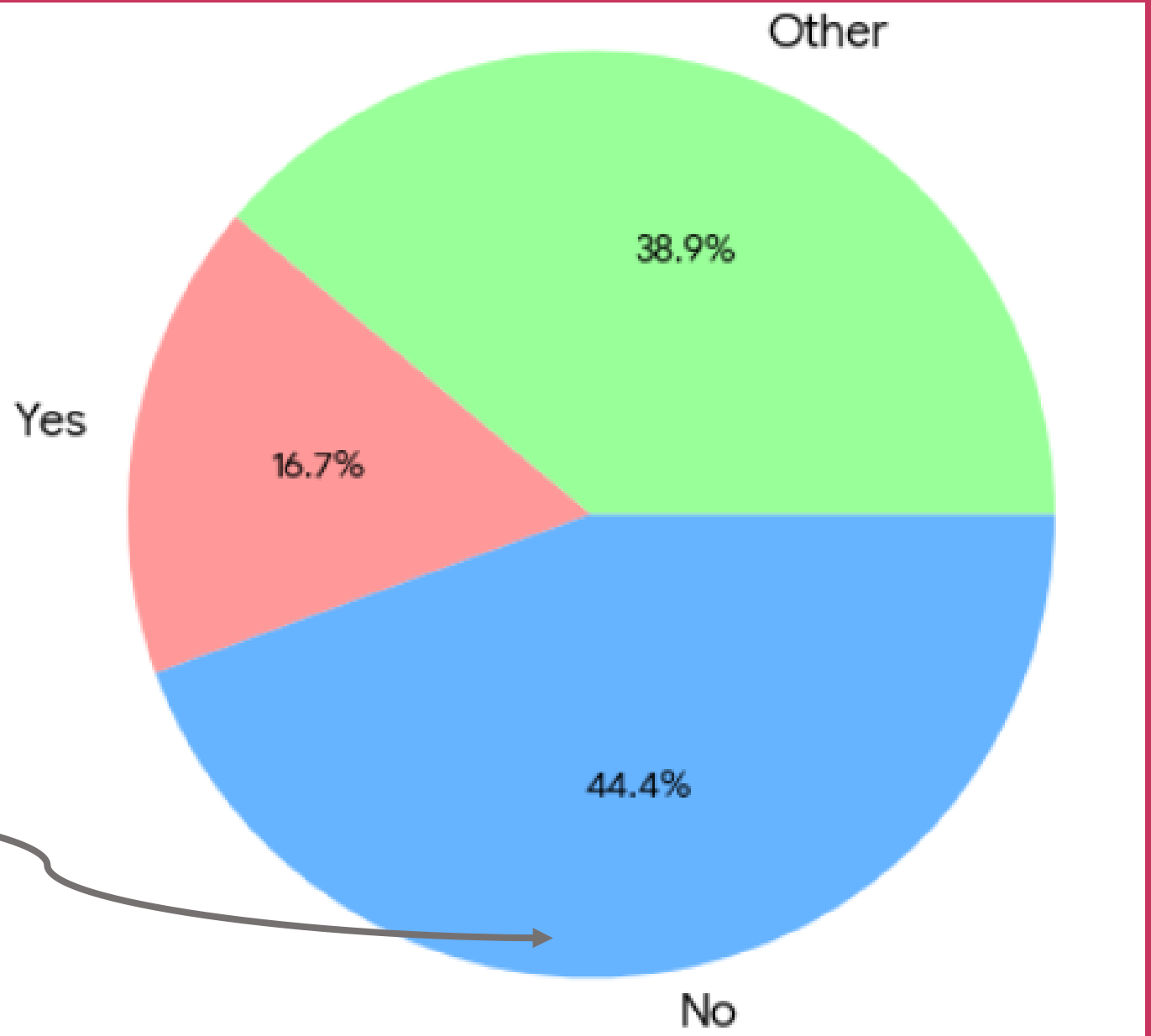
- » 1 » *unitas.*
- » $a + 1$ » *sequens a , sive a plus 1.*
- » $=$ » *est aequalis. Hoc ut novum signum considerandum est, etsi logicae signi figuram habeat.*

Is 0 a Natural Number?



Peano in 1899

“No”



ARITHMETICES PRINCIPIA.

§ 1. De numeris et de additione.

Explicationes.

Signo N significatur *numerus (integer positivus)*.

- » 1 » *unitas.*
- » $a + 1$ » *sequens a , sive a plus 1.*
- » $=$ » *est aequalis. Hoc ut novum signum considerandum est, etsi logicae signi figuram habeat.*

Axiomata.

1. $1 \in N.$
2. $a \in N. \supset a = a.$
3. $a, b, c \in N. \supset a = b. =. b = a.$
4. $a, b \in N. \supset a = b. b = c. \supset a = c.$
5. $a = b. b \in N. \supset a \in N.$
6. $a \in N. \supset a + 1 \in N.$
7. $a, b \in N. \supset a = b. =. a + 1 = b + 1.$
8. $x \in N. \supset x + 1 - = 1.$
9. $k \in K. \therefore 1 \in k. \therefore x \in N. x \in k. \supset x + 1 \in k. :: \supset N \supset k.$

Definitiones.

10. $2 = 1 + 1; 3 = 2 + 1; 4 = 3 + 1; \text{ etc.}$

Constructive Definitions

Rule 1: 0 is a Natural Number

Rule 2: If n is a Natural Number,
 $S(n)$ is a Natural Number.

0 $S(0)$ $S(S(0))$

How should we define equality for Natural Numbers?

Rule 1: 0 is a Natural Number

Rule 2: If n is a Natural Number,
 $S(n)$ is a Natural Number.

Defining Equality

Rule 1: 0 is a Natural Number

Rule 2: If n is a Natural Number,
 $S(n)$ is a Natural Number.

Defining Equality

Definition. For any two Natural Numbers, n and m , we define equality ($n = m$) as:

(1) if n is 0 : $n = m$ iff m is 0 .

(2) otherwise, n is $S(p)$ for some Natural Number p .

~~(2a)~~ If m is 0 , n is **not equal** to m .

(2b) Otherwise, m is $S(q)$ for some Natural Number q . Then, $n = m$ iff $p = q$.

base case

2 2
Prove $S(S(0)) = S(S(0))$.

Definition. For any two natural numbers, n and m , we define equality ($=$) as:

- (1) if n is 0 : $n = m$ iff m is 0 .
- (2) otherwise, n is $S(p)$ for some natural number p .
 - (2a) If m is 0 , n is not equal to m .
 - (2b) Otherwise, m is $S(q)$ for some natural number q , then $n = m$ iff $p = q$.

- 1) By definition of equality (2b), $S(S(0)) = S(S(0))$ iff $S(0) = S(0)$,
- 2) By definition of equality (2b), $S(0) = S(0)$ iff $0 = 0$.
- 3) By definition of equality (1), $0 = 0$.
- 4) Thus, $S(0) = S(0)$ (from step 2), and $S(S(0)) = S(S(0))$ (from step 1).

Quod Erat Demonstrandum

What kind of proof is this?

induction

Rule 1: 0 is a Natural Number

Rule 2: If n is a Natural Number,
 $S(n)$ is a Natural Number.

Is there any Natural Number m such that $S(m) = 0$?

No, $m = -1$
~~is~~ not \mathbb{N}
prove?

Rule 1: 0 is a Natural Number

Rule 2: If n is a Natural Number,
 $S(n)$ is a Natural Number.

Is there any Natural Number m such that $S(m) = 0$? **No, by axiom.**

Axiomata.

1. $1 \in \mathbb{N}.$
2. $a \in \mathbb{N}. \supset . a = a.$
3. $a, b, c \in \mathbb{N}. \supset : a = b. = . b = a.$
4. $a, b \in \mathbb{N}. \supset : a = b. b = c : \supset . a = c.$
5. $a = b. b \in \mathbb{N} : \supset . a \in \mathbb{N}.$
6. $a \in \mathbb{N}. \supset . a + 1 \in \mathbb{N}.$
7. $a, b \in \mathbb{N}. \supset : a = b. = . a + 1 = b + 1.$
8. $a \in \mathbb{N}. \supset . a + 1 \neq 1.$
9. $k \in \mathbb{K}.. 1 \in k.. x \in \mathbb{N}. x \in k : \supset . x + 1 \in k : : \supset . \mathbb{N} \supset k.$



In mathematics or logic, an *axiom* is an **unprovable** rule or first principle accepted as **true** because it is self-evident or particularly useful.

Using the Definition

Rule 1: 0 is a Natural Number

Rule 2: If n is a Natural Number,
 $S(n)$ is a Natural Number.

Notation: The set of Natural Numbers is denoted as \mathbb{N} .

Induction



Aqua Claudia, finished by Emperor [Claudius](https://en.m.wikipedia.org/wiki/Claudius) (41–54 AD) in 52 AD. https://en.m.wikipedia.org/wiki/Aqua_Claudia

Principle of Induction

Suppose that X is a subset of \mathbb{N}
that satisfies these two properties:

(1) $0 \in X$

(2) if $n \in X$, then $S(n) \in X$.

Then, $X = \mathbb{N}$.

Is this an axiom? Logic rule? Theorem?

Principle of Induction

Suppose that X is a subset of \mathbb{N} that satisfies these two properties:

(1) $0 \in X$

(2) if $n \in X$, then $S(n) \in X$.

Then, $X = \mathbb{N}$.

Whoa! We've just used sets, subset, element of, set equality, etc. all without defining them!

Suppose that X is a subset of \mathbb{N}
that satisfies these two properties:

(1) $0 \in X$

(2) if $n \in X$, then $S(n) \in X$.

Then, $X = \mathbb{N}$.

Principle of Induction

Principle of Induction: Suppose that X is a subset of \mathbb{N} that satisfies these two properties: (1) $0 \in X$ (2) if $n \in X$, then $S(n) \in X$. Then, $X = \mathbb{N}$.

Proof: 1. Define $Y = \mathbb{N} \setminus X$. (All elements of \mathbb{N} that are not in X .)

2. For contradiction, assume Y is non-empty.

3. Then, Y has some smallest element z .

(Note: this is non-obvious – using well-ordering principle.)

4. By property (1), z cannot be 0 .

5. Thus, $z = S(p)$ for some $p \in \mathbb{N}$.

Why?

$z \neq 0$ by axiom 8

6. Since p is smaller than z , it cannot be in Y , so must be in X . $p \in X$

7. But, this contradicts property (2), since if $p \in X$, then $S(p) \in X$.

8. Thus, the assumption that Y is non-empty is invalid.

9. Hence, Y is empty, and $\mathbb{N} = X$.

Proof by Induction

Principle of Induction: Suppose that X is a subset of \mathbb{N} that satisfies these two properties: (1) $0 \in X$ (2) if $n \in X$, then $S(n) \in X$. Then, $X = \mathbb{N}$.

Proof by Induction: For any predicate $P(\mathbb{N})$, showing (1) $P(0)$ and (2) if for any $n \in \mathbb{N}$ if $P(n)$ then $P(S(n))$ proves P holds for all \mathbb{N} .

Sets

Defining Sets

Base Clause:

The empty set (denoted as \emptyset) is a Set.

Inductive Clause:

If S is a Set and x is anything, $S \cup \{x\}$ is a Set.

$\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \dots$

Constructive Definitions

Rule 1: 0 is a Natural Number

Rule 2: If n is a Natural Number, $S(n)$ is a Natural Number.

Rule 1: \emptyset is a Set.

Rule 2: If S is a set and x is anything, $S \cup \{x\}$ is a set.

0 , \emptyset , S , \cup , $+$, $\{\}$, etc. are all just symbols: their meaning comes from the definitions we agree to give them.

✧ Question: Is this a good definition?

Rule 1: \emptyset is a Set.

Rule 2: If S is a set and x is anything, $S \cup \{x\}$ is a set.

$\{0, 1, 2, 3, \dots, 100\}$

Q: Is (the whole of) the natural numbers a set?

Maybe not

Not by above

Constructive definitions are **finite**.

5.4. Sets

Python also includes a data type for *sets*. A set is an unordered collection with no duplicate elements. Basic uses include membership testing and eliminating duplicate entries. Set objects also support mathematical operations like union, intersection, difference, and symmetric difference.

Curly braces or the `set()` function can be used to create sets. Note: to create an empty set you have to use `set()`, not `{}`; the latter creates an empty dictionary, a data structure that we discuss in the next section.

Here is a brief demonstration:

```
>>> basket = {'apple', 'orange', 'apple', 'pear', 'orange', 'banana'}
>>> print(basket)                # show that duplicates have been removed
{'orange', 'banana', 'pear', 'apple'}
>>> 'orange' in basket           # fast membership testing
True
>>> 'crabgrass' in basket
False
```

(Often) Good in programming
Not so in this course!

```
{ 'r', 'd', 'b' }
>>> a | b                        # Letters in a or b or both
{'a', 'c', 'r', 'd', 'b', 'm', 'z', 'l'}
>>> a & b                        # Letters in both a and b
{'a', 'c'}
>>> a ^ b                        # Letters in a or b but not both
{'r', 'd', 'b', 'm', 'z', 'l'}
```


3. Set

A **set** is a value whose only property is having other values as its **members**. The most common representation of a set is as its members written between braces. The members of a set have no position, order, number of times appearing in the set, or any other properties beyond being a member of the set.

The number of distinct members of a set is its **cardinality** and is denoted $|S|$. A set with cardinality 1 is a **singleton set**.

A set can be defined based on any **predicate** using **set-builder notation**: $\{f(x) \mid P(x)\}$ is the set of all $f(x)$ where $P(x)$ is true.

Example —

$\{x + 2 \mid 1 < x \leq 2\}$ is the set of all numbers greater than 3 and no greater than 4.

$x \in \{x \mid P(x)\}$ is a long way of writing $P(x)$.

$\{x \mid x \in S\}$ is a long way of writing S .

set-builder predicate

Rule 1: 0 is a Natural Number

Rule 2: If n is a Natural Number,
 $S(n)$ is a Natural Number.

Practice: Defining +

Rule 1: 0 is a Natural Number

Rule 2: If n is a Natural Number,
 $S(n)$ is a Natural Number.

Practice: Defining $+$

Definition. The *sum* of two natural numbers a and b (denoted as $a + b$) is defined as:

- (1) If a is 0 , the $a + b$ sum is b .
- (2) Otherwise, a is $S(p)$ for some Natural Number p and $a + b$ is $S(p + b)$.

$$0 + b = b$$

$$(p + 1) + b = (b + p) + 1$$

$Suc(p)$

$Suc(b+p)$

Representing the Natural Numbers

Rule 1: 0 is a Natural Number

Rule 2: If n is a Natural Number,
 $S(n)$ is a Natural Number.

How to *represent* the natural numbers?

5: $S(S(S(S(S(0))))))$
 | | | | |

Can we represent the Natural Numbers using finite binary strings?

Strings: Another set we will use time and again is

TCS, 1.4.2

$$\{0, 1\}^n = \{(x_0, \dots, x_{n-1}) : x_0, \dots, x_{n-1} \in \{0, 1\}\}$$

which is the set of all n -length binary strings for some natural number n . That is $\{0, 1\}^n$ is the set of all n -tuples of zeroes and ones. This is consistent with our notation above: $\{0, 1\}^2$ is the Cartesian product $\{0, 1\} \times \{0, 1\}$, $\{0, 1\}^3$ is the product $\{0, 1\} \times \{0, 1\} \times \{0, 1\}$ and so on.

We will write the string $(x_0, x_1, \dots, x_{n-1})$ as simply $x_0x_1 \cdots x_{n-1}$. For example,

$$\{0, 1\}^3 = \{000, 001, 010, 011, 100, 101, 110, 111\} .$$

For every string $x \in \{0, 1\}^n$ and $i \in [n]$, we write x_i for the i^{th} element of x .

We will also often talk about the set of binary strings of *all* lengths, which is

$$\{0, 1\}^* = \{(x_0, \dots, x_{n-1}) : n \in \mathbb{N}, x_0, \dots, x_{n-1} \in \{0, 1\}\} .$$

Another way to write this set is as

$$\{0, 1\}^* = \{0, 1\}^0 \cup \{0, 1\}^1 \cup \{0, 1\}^2 \cup \dots$$

or more concisely as

$$\{0, 1\}^* = \bigcup_{n \in \mathbb{N}} \{0, 1\}^n .$$

Binary Strings


Defining Binary Strings

If we want to define binary strings with a constructive recursive definition, what is the best choice for the base clause?


Defining Binary Strings

Defining Binary Strings

Base:

" " (empty) is a Binary String


Inductive:

If s is a Binary String,
both **0** s and **1** s are Binary Strings.


3 Symbols

Another way to write this set is as

$$\{0, 1\}^* = \{0, 1\}^0 \cup \{0, 1\}^1 \cup \{0, 1\}^2 \cup \dots$$

or more concisely as

$$\{0, 1\}^* = \bigcup_{n \in \mathbb{N}} \{0, 1\}^n .$$

Can all Natural Numbers be represented by Binary Strings?

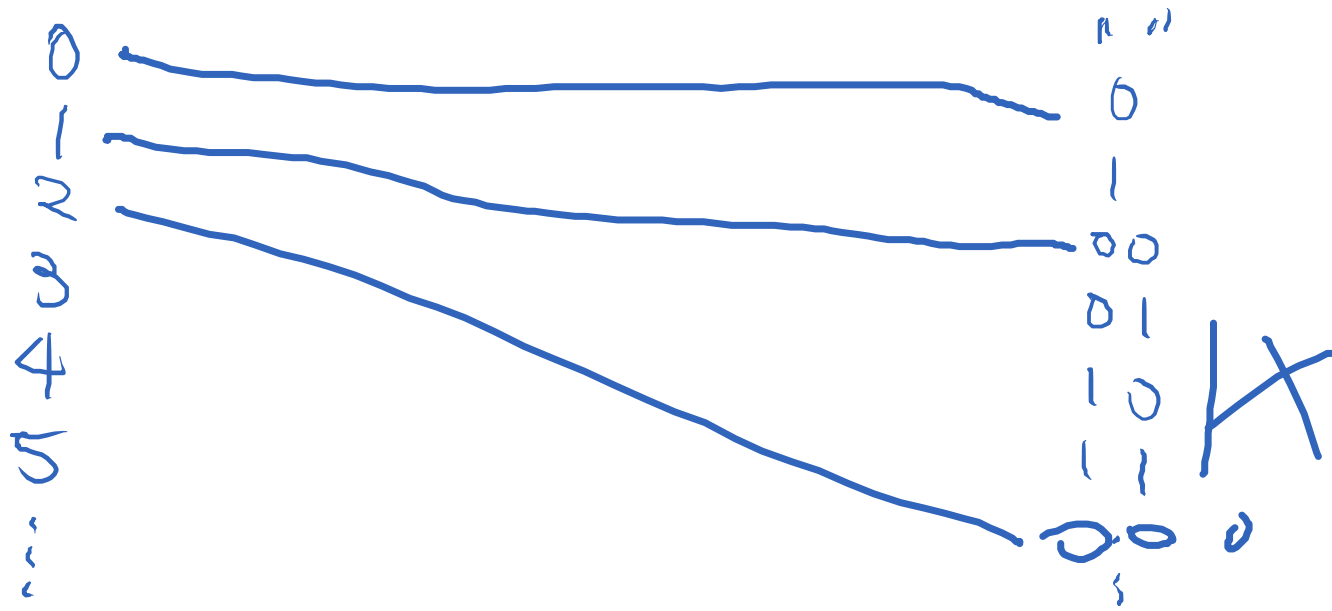
Rule 1: 0 is a Natural Number

Rule 2: If n is a Natural Number, $S(n)$ is a Natural Number.

Rule 1: "" is a Binary String

Rule 2: If s is a Binary String, both 0s and 1s are Binary Strings.

Want: each n is mapped to a unique s



Can all Binary Strings be represented by Natural Numbers?

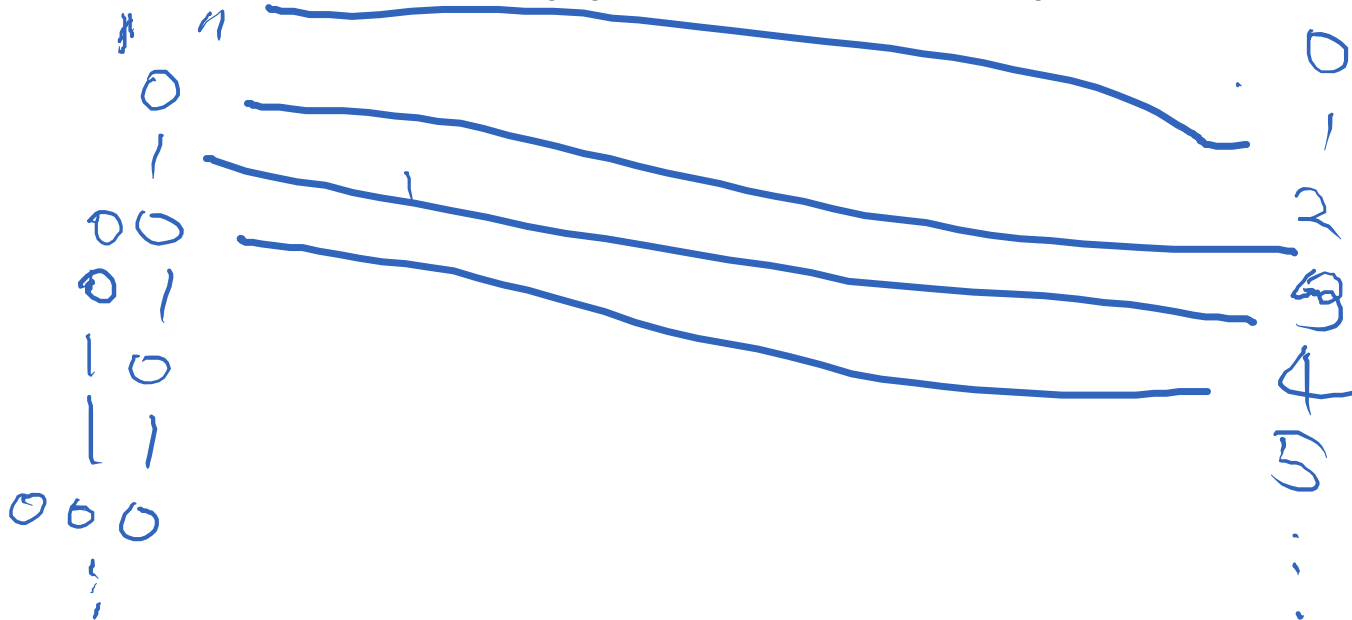
Rule 1: "" is a Binary String

Rule 2: If s is a Binary String, both $0s$ and $1s$ are Binary Strings.

Rule 1: 0 is a Natural Number

Rule 2: If n is a Natural Number, $S(n)$ is a Natural Number.

Want: each s is *mapped* to a unique n



Recap

Mathematical Definitions

What makes a good definition?

Defining the Natural Numbers

Induction Principle

Sets

Defining the binary strings

Today: Chapter 0 and 1 in the TCS book