

Class 19: Computability and Uncomputability

Theorem 1 (Universal Turing Machine (Section 9.1)). *There exists a Turing machine U such that on every string w , which represents a Turing machine TM_w , and every string $x \in \{0, 1\}^*$, it holds that $U(w, x) = \text{TM}_w(x)$.*

That is, if the machine TM_w halts on x and outputs some $y \in \{0, 1\}^$, then $U(w, x) = y$, and if TM_w does not halt on x (i.e., $\text{TM}_w(x) = \perp$) then $U(w, x) = \perp$.*

Remark: The theorem assumes a fixed representation that encodes any Turing Machine M into a unique string $w \in \{0, 1\}^*$. By TM_w , we denote the Turing Machine that is represented by w . For any $w \in \{0, 1\}^*$ such that there is no Turing Machine mapped to w by the representation, we define TM_w to be a fixed Turing Machine that ignores the input and always outputs 0.

Definition 2 (Computable functions (Section 7.1.3)). *Let $F : \{0, 1\}^* \rightarrow \{0, 1\}$ be a (total) function and let M be a Turing machine. We say that M computes F if for every $x \in \{0, 1\}^*$, $M(x) = F(x)$.*

We say that a function F is computable if there exists a Turing machine M that computes it.

An Uncomputable Function

We define the Boolean function $NSA : \{0, 1\}^* \rightarrow \{0, 1\}$ to be

$$NSA(x) = \begin{cases} 0 & \text{if } \text{TM}_x(x) = 1 \\ 1 & \text{otherwise.} \end{cases}$$

NSA stands for Not-Self-Accepting, as $\text{TM}_x(x)$ is self-accepting (which is denoted as F^* in Section 9.2). We prove that NSA is uncomputable.

Proof. Assume for contradiction, NSA is computable. By definition of computable, there exists a Turing Machine M such that $M(x) = NSA(x)$ for all x . Let $w \in \{0, 1\}^*$ be the string representing M , so that $\text{TM}_w = M$. Consider the outcome of $U(w, w)$, where U is the universal Turing Machine. We have

$$U(w, w) = \text{TM}_w(w) = M(w) = NSA(w)$$

by definition of universal Turing Machine, then by representation TM, then by definition of M . However, $NSA(w) = 0$ if and only if $\text{TM}_w(w) = 1$ by definition of NSA , which contradicts the above equation. \square