Kolmogorov Complexity

CS 3120 - April 8, 2025

Recall

A *Turing Machine,* is defined by (Σ, k, δ):

$\boldsymbol{k} \in \mathbb{N}$: a finite number of states

Σ : alphabet $-$ finite set of symbols

$$\Sigma \supseteq \{0, 1, \rhd, \emptyset\}$$

$\boldsymbol{\delta}$: transition function

$$\boldsymbol{\delta}: [k] \times \Sigma \rightarrow [k] \times \Sigma \times \{\mathbf{L}, \mathbf{R}, \mathbf{S}, \mathbf{H}\}$$

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$$|\, TMs \,| = |\, \text{finite binary strings} \,| = |\, \mathbb{N} \,|$$

What is information?

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00000000000000000000

10101111010101111100

What is a computable random number?

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0.1010010001000010000010000001

0.15264839763039644951841857640

What is an incompressible string?

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1010101010101010

0110011101010001

How much information is in a string?

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100100100100100100100100100

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How much information is in a string?

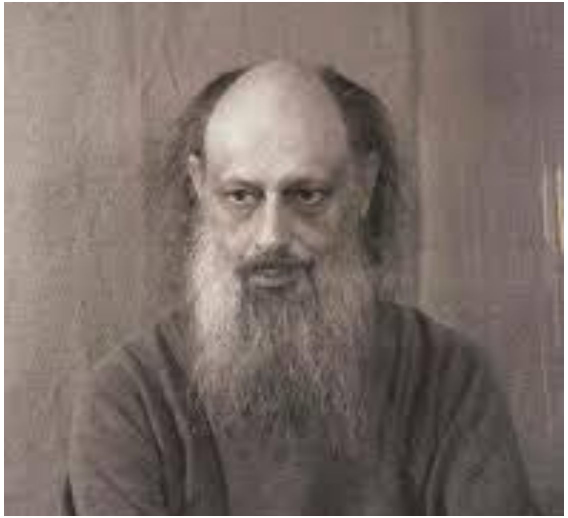
23571113171923293137414347 53
596167717379838997

How much information is in a string?

383862748878325473579680183
468291898745981708710670140
958198041893735035359221176

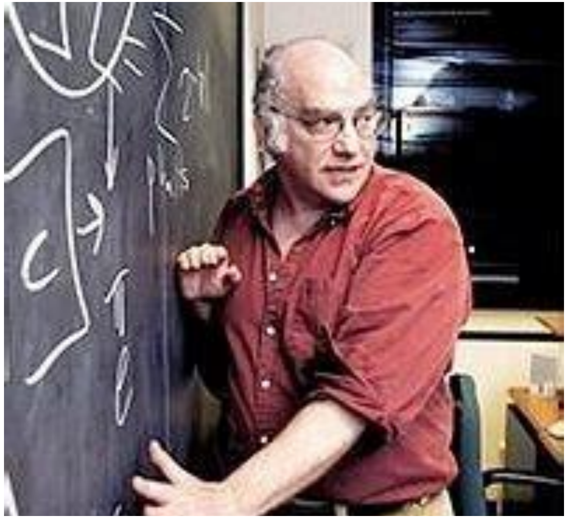
Kolmogorov Complexity

Origins of Kolmogorov Complexity



Ray Solomonoff (1964)

Andrey Kolmogorov (1965)



Gregory Chaitin (1969)

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Kolmogorov Complexity of x, denoted $K_U(x)$

$$K_U(x) = \min_n \{ | < M_n > | : U \text{ simulates } M_n \text{ and outputs } x \}$$

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K(x) is the length of the shortest description of x

Kolmogorov Theory Applications

Mathematics: probability theory, logic

Physics: chaos, thermodynamics

Computer Science: average case analysis, inductive inference and learning, shared information between documents, data mining and clustering, incompressibility method

Misc: randomness, inference, complex systems, sequence similarity

Information Theory: information in individual objects, information distance

Invariance Theorem

Kolmogorov Complexity is robust.

The choice of universal Turing machine only affects complexity by an additive constant.

→ All encoding methods are equivalent up to a constant.

Invariance Theorem

There exists a Turing machine U such that for all Turing machines M_n, there exists a constant c_n such that for all strings $x \in \{0,1\}^*$,

$$K_U(x) \leq K_{M_n}(x) + c_n$$

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The constant c_n depends only on n, not on x. $\square$

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3. $K(n!) \leq O(\log n)$
4. For all x, $K(x) \leq |x| + O(1)$
5. $K(xy) \leq K(x) + K(y) + O(\log(\min\{K(x), K(y)\})$

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Incompressible strings have properties similar to random strings.

There are infinitely many incompressible strings.

There are infinitely many incompressible binary strings. That is, for any constant c, there exist infinitely many strings $x \in \{0,1\}^*$ such that $K(x) \geq |x| - c$ where $K(x)$ denotes the Kolmogorov complexity of x.

Fix constant $c > 0$ and $n \in \mathbb{N}$ and consider the set of all length n binary strings. There are exactly 2^n such strings.

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So the number of strings x of length n such that $K(x) \geq n - c$ is at least:

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Thus, for any constant c, there are infinitely many incompressible strings. $\square$