Homework 1

Due: **10:00pm, Friday, January 30**

This problem set focuses on understanding induction and uncountability (Chapter 1-2 in TCS). Write your answers in the hw1.tex LaTeX template. You will submit your solutions in GradeScope as a PDF file with your answers to the questions in this template.

Collaboration Policy: You may discuss the problems with anyone you want. You are permitted to use any resources you find for this assignment **other than *solutions* from previous/concurrent CS3120 courses**. You must write up your own solutions and understand everything in them, and submit only your own work. You should note in the *Collaborators and Resources* box below the people you collaborated with and any external resources you used (you do not need to list resources you used for help with LaTeX).

Collaborators and Resources: TODO: replace this with your collaborators and resources (if you did not have any, replace this with *None*)

To do this assignment:

1. Open this read-only Overleaf project located at https://www.overleaf.com/read/vvkpffrxbvxn#4623fe, then select the "File" button at the top-left, and then select "Make a copy". You will have an opportunity to rename the project, and then Overleaf will create a new copy of the project which you can edit.
2. Open your copy of the project and in the left side of the browser, you should see a file directory containing hw1.tex. Click on hw1.tex to see the LaTeX source for this file, and enter your solutions in the marked places. (You will also see the uvatoc.sty file, a “style” file that defines some useful macros. You are welcome to look at this file but should not need to modify it.)
3. The first thing you should do in hw1.tex is set up your name as the author of the submission by replacing the line, `\submitter{TODO: your name}`, with your name and UVA id, e.g., `\submitter{Wei-Kai Lin (twc7zv)}`.
4. Write insightful and clear answers to all of the questions. As typical people, we prefer short, precise, and comprehensive answers.
5. There are *optional* problems. There are no points, but if you wrote your solutions, we will tell you how you did.
6. Before submitting your hw1.pdf file, also remember to:
   - List your collaborators and resources, replacing the TODO in `\collaborators{TODO: replace ...}` with your collaborators and resources. (Remember to update this before submitting if you work with more people.)

Problem 1 (6pt each) *English sentences is countably infinite.*

Let $S = \Sigma^* = \{(a_0, a_1, a_2, \ldots, a_n) \mid n \in \mathbb{N}, a_i \in \Sigma\}$, where $\Sigma = \{$A, B, C, ..., Z, ␣$\}$ is the set of English alphabets and blank space (total 27 symbols).

1. Prove that S is countable, that is, there is a subjective function from $\mathbb{N}$ to S.

   Note: Instead of proving a subjective function $\mathbb{N}$ to S, we often recommend proving that there is an injective (total) function from S to $\mathbb{N}$, which implies a subjective function from $\mathbb{N}$ to S (you don't need to prove this implication, but it is good to know why).

2. Prove that S is infinite, that is, there exists a strict subset $T \subsetneq S$ such that there is a bijection from S to T.

Answer:

1.

2.

Problem 2 (6pt each) *Three-dimensional grids is countable.*

Here is a constructive definition of the set of all integers, $\mathbb{Z}$.

Definition 1 (Integers, $\mathbb{Z}$) *For any natural number $n \in \mathbb{N}$, n is an integer; moreover, if $n > 0$, $-n$ is also an integer.*

1. Prove that there is a bijective (one-to-one and onto) function from $\mathbb{N} \to \mathbb{Z}$. Note: This implies that $|\mathbb{Z}| = |\mathbb{N}|$ and thus $|\mathbb{Z}|$ is countable.
2. Prove that there is a surjective function from $\mathbb{N} \to \mathbb{N}^3$. Note that to prove subjectivity, each element in $\mathbb{N}$ must be mapped to at most one element in $\mathbb{N}^3$, and each element in $\mathbb{N}^3$ must be mapped from at least one element in $\mathbb{N}$.

   Hint: Let $C(n)$ be the number of points $(x_1, x_2, x_3) \in \mathbb{N}^3$ such that $x_1, x_2, x_3 \leq n$ for any $n \in \mathbb{N}$. Then, $C(n+1) - C(n)$ is the quantity of distinct natural numbers we need for the points for each $n + 1$.
3. The set of integer points in 3-dimension is the set $\mathbb{Z}^3 = \{(x_1, x_2, x_3) \mid x_1, x_2, x_3 \in \mathbb{Z}\}$. Prove that $\mathbb{Z}^3$ is countable, that is, there is a subjective function from $\mathbb{N}$ to $\mathbb{Z}^3$. You can use the results from the previous steps.

Answer:

1.

2.

3.

Problem 3 (optional) *Map lists of integers to a number.*

For every set S, the set S^* is defined as the set of all finite sequences of members of S (i.e., $S^* = \{(x_0, ..., x_{n-1}) \mid n \in \mathbb{N}, \forall_{i \in [n]} x_i \in S\}$). Prove that $|\mathbb{Z}^*| \leq |\mathbb{N}|$ where $\mathbb{Z}$ is the set of all integers $\{..., -3, -2, -1, 0, +1, +2, +3, ...\}$. That is, to show a surjective function from $\mathbb{N}$ to $\mathbb{Z}^*$.

Note: It is easy to show that $|\mathbb{N}| \leq |\mathbb{Z}^*|$, and together with the above, it follows that $|\mathbb{Z}^*| = |\mathbb{N}|$ by (Cantor-)Schröder–Bernstein theorem.

Answer:

Problem 4 (optional) *The number of binary tree leaves*

Prove that any binary tree of height $h \in \mathbb{N}$ has at most 2^h leaves.

Note: We haven't defined a *binary tree* (and the textbook doesn't). An adequate answer to this question will use the informal understanding of a binary tree which we expect you have entering this class, but an excellent answer will include a definition of a binary tree and connect your proof to that definition.

Answer:

Do not write anything on this page; leave this page empty.

This is the end of the problems for HW1. Remember to follow the last step in the directions on the first page to prepare your PDF for submission.

