Homework 5

Response by: **TODO: replace this with your name (and computing id)**

Due: **10:00pm, Friday, March 13**

This problem set focuses on circuits (Chapters 3–5 in TCS). Write your answers in the hw5.tex LaTeX template. You will submit your solutions in GradeScope as a PDF file with your answers to the questions in this template.

Collaboration Policy: You may discuss the problems with anyone you want. You are permitted to use any resources you find for this assignment **other than *solutions* from previous/concurrent CS3120 courses**. You must write up your own solutions and understand everything in them, and submit only your own work. You should note in the *Collaborators and Resources* box below the people you collaborated with and any external resources you used (you do not need to list resources you used for help with LaTeX).

Collaborators and Resources: TODO: replace this with your collaborators and resources (if you did not have any, replace this with *None*)

To do this assignment:

1. Open this read-only Overleaf project located at https://www.overleaf.com/read/tnysrnvrxytm#16a3cb, then select the "File" button at the top-left, and then select "Make a copy". You will have an opportunity to rename the project, and then Overleaf will create a new copy of the project which you can edit.

2. Open your copy of the project and in the left side of the browser, you should see a file directory containing hw5.tex. Click on hw5.tex to see the LaTeX source for this file, and enter your solutions in the marked places. (You will also see the uvatoc.sty file, a "style" file that defines some useful macros. You are welcome to look at this file but should not need to modify it.)

3. The first thing you should do in hw5.tex is set up your name as the author of the submission by replacing the line, `\submitter{TODO: your name}`, with your name and UVA id, e.g., `\submitter{Wei-Kai Lin (twc7zv)}`.

4. Write insightful and clear answers to all of the questions. As typical people, we prefer short, precise, and comprehensive answers.

5. There are *optional* problems. There are no points, but if you wrote your solutions, we will tell you how you did.

6. Before submitting your hw5.pdf file, also remember to:

   – List your collaborators and resources, replacing the TODO in `\collaborators{TODO: replace ...}` with your collaborators and resources. (Remember to update this before submitting if you work with more people.)

Problem 1 (6pt each) *Constant Functions*

In the proof of circuit size hierarchy [Bar23, Theorem 5.5], we supposed that for all natural numbers $n > 0$, the function $f : \{0,1\}^n \to \{0,1\}$ can be computed by a NAND circuit of size $< 10n$. We formally prove it in this problem.

(a) When $n = 1$, write a NAND circuit C_1 such that for all $x \in \{0,1\}^1$, $C_1(x) = 0$.

(b) Suppose that for some $n \geq 1$, there exists a NAND circuit C_n such that for all $x \in \{0,1\}^n$, $C_n(x) = 0$. Write a NAND circuit C_{n+1} such that for all $x \in \{0,1\}^{n+1}$, $C_{n+1}(x) = 0$, where C_{n+1} may use C_n.

(c) For all n, calculate an upper bound on the number of NAND gates in your circuit C_n, where the upper bound shall be $< 10n$.

Answer:

(a)

(b)

(c)

Problem 2 (12pt) *Random Functions are Hard [Bar23, Exercise 5.8]*

Suppose $n > 1000$ in this problem. Suppose that we choose a function $F : \{0,1\}^n \to \{0,1\}$ at random; that is, choosing for every $x \in \{0,1\}^n$ the value $F(x)$ to be the result of tossing an independent unbiased coin. Prove that the probability that there is a $2^n/(1000n)$-line program (namely, a NAND-straightline program of $2^n/(1000n)$ many lines) that computes F is at most 2^{-100}.

(Hint: Exercise 5.8 gives a hint if you get stuck.)

Note: This is another *existential* proof showing that there exists a hard-to-compute function. It is existential because we do not know whether the sampled function F satisfies the above circuit size.

Answer:

Problem 3 (Optional) *Fraction of Easy Functions*

Consider the previous Problem 2 again using the same n, F, and the threshold of $t(n)$-line programs, where $t(n) = 2^n/(1000n)$. The previous problem essentially shows that $|SIZE_{n,1}(t(n))| \leq 2^{-100} \cdot 2^{2^n}$, where 2^{2^n} is the number of distinct F, and $SIZE(t(n))$ is the set of "easy" functions that can be computed in $t(n)$ lines. How tight is the bound 2^{-100}? Try to find a function $\epsilon(n)$ such that

$$|SIZE_{n,1}(t(n))| \leq \epsilon(n) \cdot 2^{2^n}$$

for all $n > 1000$. Ideally, we want $\epsilon(n)$ as small as possible. Preferably, $\epsilon(n)$ decreases as n increases, and $\epsilon(n) < 2^{-100}$ for sufficiently large n.

Answer:

Problem 4 (Optional) *Another Size Hierarchy*

Recall that $SIZE_{n,1}(s)$ denotes the set of functions $f : \{0,1\}^n \to \{0,1\}$ such that f can be computed by a NAND circuit of s gates. Is the following statement true? Prove it (which can be based on any textbook theorem) or refute it (often by providing a counterexample).

> There exists a constant $C \in \mathbb{N}$ such that for all sufficiently large $n \in \mathbb{N}$ and for all $10n < s < \frac{2^n}{10n}$, there exists a function $g : \{0,1\}^n \to \{0,1\}$ such that $g \in SIZE_{n,1}(C \cdot s) \setminus SIZE_{n,1}(s)$.

Hint: There is a similar problem in the textbook [Bar23, Exercise 5.5], which can be helpful.

Answer:

Do not write anything on this page; leave this page empty.

This is the end of the problems for HW5. Remember to follow the last step in the directions on the first page to prepare your PDF for submission.

References

[Bar23] Boaz Barak. *Introduction to Theoretical Computer Science*. https://introtcs.org/public/. 2023.