Homework 8

Response by: **TODO: replace this with your name (and computing id)**

Due: **10:00pm, Friday, April 17**

This problem set focuses on complexity and polynomial-time reductions, Chapters 12–15 in TCS Textbook). Write your answers in the hw8.tex LaTeX template. You will submit your solutions in GradeScope as a PDF file with your answers to the questions in this template.

> **Collaboration Policy:** You may discuss the problems with anyone you want. You are permitted to use any resources you find for this assignment **other than *solutions* from previous/concurrent CS3120 courses**. You must write up your own solutions and understand everything in them, and submit only your own work. You should note in the *Collaborators and Resources* box below the people you collaborated with and any external resources you used (you do not need to list resources you used for help with LaTeX).

> **Collaborators and Resources:** TODO: replace this with your collaborators and resources (if you did not have any, replace this with *None*)

To do this assignment:

1. Open this read-only Overleaf project located at https://www.overleaf.com/read/qrhqmpfgdhzq#792750, then select the "File" button at the top-left, and then select "Make a copy". You will have an opportunity to rename the project, and then Overleaf will create a new copy of the project which you can edit.
2. Open your copy of the project and in the left side of the browser, you should see a file directory containing hw8.tex. Click on hw8.tex to see the LaTeX source for this file, and enter your solutions in the marked places. (You will also see the uvatoc.sty file, a "style" file that defines some useful macros. You are welcome to look at this file but should not need to modify it.)
3. The first thing you should do in hw8.tex is set up your name as the author of the submission by replacing the line, `\submitter{TODO: your name}`, with your name and UVA id, e.g., `\submitter{Wei-Kai Lin (twc7zv)}`.
4. Write insightful and clear answers to all of the questions. As typical people, we prefer short, precise, and comprehensive answers.
5. There are *optional* problems. There are no points, but if you wrote your solutions, we will tell you how you did.
6. Before submitting your hw8.pdf file, also remember to:
   - List your collaborators and resources, replacing the TODO in `\collaborators{TODO: replace ...}` with your collaborators and resources. (Remember to update this before submitting if you work with more people.)

Problem 1 (14pt). *Polynomial-Time Reductions*

For each sub-problem, indicated if the state proposition is **True** or **False**, and provide a brief (one or two sentences) justification for your answer.

We use the notations in the book: let $F, G : \{0,1\}^* \to \{0,1\}^*$. We say that F reduces to G, denoted by $F \leq_p G$, if there is a polynomial-time computable $R : \{0,1\}^* \to \{0,1\}^*$ such that for every $x \in \{0,1\}^*$, $F(x) = G(R(x))$.

(a) $F \leq_p G$ and $G \in \mathbf{P}$ implies $F \in \mathbf{P}$.

(b) $F \leq_p G$ and $F \in \mathbf{P}$ implies $G \in \mathbf{P}$.

(c) $F \leq_p G$ and $G \in \mathbf{EXP}$ implies $F \in \mathbf{EXP}$.

(d) $F \leq_p G$ and $G \leq_p F$ implies $F \in \mathbf{P}$.

(e) $F \leq_p G$ and G is computable implies F is computable.

(f) $F \leq_p G$ and F is computable implies G is computable.

(g) $F \leq_p G$ and $G \leq_p F$ implies $F = G$.

Answer:

(a) [T] [F] Reason:

(b) [T] [F] Reason:

(c) [T] [F] Reason:

(d) [T] [F] Reason:

(e) [T] [F] Reason:

(f) [T] [F] Reason:

(g) [T] [F] Reason:

Problem 2 (16pt). *Jeffersonian Paths*

The Hamlitonian Path problem (not named after Alexander), defined below, is known to be $\mathbf{NP}$-complete.

> *HAMILTONIAN PATH*
> **Input:** A description of an undirected, finite graph, $G = (V, E)$.
>
> **Output:** **True** if there exists a path in G that visits each vertex in V exactly once; otherwise **False**.

Despite his declarations to the contrary, Jefferson does not consider all vertices equal, and defines the *JEFFERSONIAN PATH* problem as:

> *JEFFERSONIAN PATH*
> **Input:** A description of an undirected, finite graph, $G = (V, E)$, and a partitioning of V into two subsets V_1 and V_2 such that $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \emptyset$.
>
> **Output:** **True** if there exists a path in G that visits each vertex in V exactly once, where all vertices in V_1 are visited before any vertex in V_2; otherwise **False**.

Prove *JEFFERSONIAN PATH* is not easier than *HAMILTONIAN PATH*; that is, show a reduction *HAMILTONIAN PATH* $\leq_p$ *JEFFERSONIAN PATH*. Below is a template for your answer, but you do not need to follow it if you have a better reduction.

Answer:

(a) Write which problem G is assumed to be solvable in polynomial time.

(b) Write which problem F we aim to solve in polynomial time.

(c) Write the reduction R as a algorithm.

(d) Argue that for every $x \in \{0, 1\}^*$, $F(x) = G(R(x))$.

Problem 3 (Optional). *Jeffersonian Paths (continued)*

(a) Similar to the previous problem, but we modify the definition of *JEFFERSONIAN PATH* so that input V_1 is not an empty set (thus consists of at least one vertex). Show the reduction *HAMILTONIAN PATH* $\leq_p$ *JEFFERSONIAN PATH*.

(b) Similar to the previous problem, but we modify the definition of *JEFFERSONIAN PATH* so that input V_1 consists of exactly half vertices, i.e., $|V_1| = \lfloor |V|/2 \rfloor$. Show the reduction *HAMILTONIAN PATH* $\leq_p$ *JEFFERSONIAN PATH*.

Answer:

Problem 4 (Optional). *Planar Graphs*

Let $PLANARMATRIX : \{0,1\}^* \to \{0,1\}$ be the function that on input an adjacency matrix A outputs 1 if and only if the graph represented by A is *planar* (that is, can be drawn on the plane without edges crossing one another). For this question, you can use without proof the fact that $PLANARMATRIX \in \mathbf{P}$. Prove that $PLANARLIST \in \mathbf{P}$ where $PLANARLIST : \{0,1\}^* \to \{0,1\}$ is the function that on input an adjacency list L outputs 1 if and only if L represents a planar graph.

Answer:

Problem 5 (Optional). *Balanced Max Cut*

For any (simple) graph $G = (V, E)$, any a subset $S \subseteq V$ and its complement $\bar{S} = V \setminus S$ is called a *cut*, and we say that S cuts c edges iff there are c edges between S and $\bar{S}$. Let $MAXCUT : \{0,1\}^* \to \{0,1\}$ be the function:

$$MAXCUT(G,k) = \begin{cases} 1 & \text{if there exist a cut } S \text{ that cuts at least } k \text{ edges,} \\ 1 & \text{otherwise.} \end{cases}$$

Let $BMC : \{0,1\}^* \to \{0,1\}$ be the function, where BMC stands for Balanced Max Cut:

$$BMC(G,k) = \begin{cases} 1 & \text{if there exist a cut } S \text{ that cuts at least } k \text{ edges and consists of exactly } \lfloor |V|/2 \rfloor \text{ vertices,} \\ 1 & \text{otherwise.} \end{cases}$$

Prove that $MAXCUT \leq_p BMC$. Note: This is Exercise 15.5. Because $MAXCUT$ is NP-complete, the above reduction proves that BMC is NP-hard.

Answer:

Do not write anything on this page; leave this page empty.

This is the end of the problems for HW8. Remember to follow the last step in the directions on the first page to prepare your PDF for submission.